3.12

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$\mathcal{A}$: abelian cat, $K(\mathcal{A})$, $Q = \{\text{quasi-isom}\}$.

§5.1

Recall: Q is a saturated compatible multi sys. $\varphi(Q) = \{\text{acyclic cpx}\}$.

Def: $D(\mathcal{A}) = Q^{-1}K(\mathcal{A})$, called the derived cat of $\mathcal{A}$.

Similar, $D^b(\mathcal{A}) := Q_b^{-1}K^b(\mathcal{A})$ bounded D.C.

$D^-(\mathcal{A}) := Q_-^{-1}K^-(\mathcal{A})$ upper bounded

$D^+(\mathcal{A}) := Q_+^{-1}K^+(\mathcal{A})$ lower bounded.

$D^*(\mathcal{A})$. $* \in \{\text{null}, -, +, b\}$.

localization functor. $F^*: K^*(\mathcal{A}) \longrightarrow D^*(\mathcal{A})$.

obj, mor & D.T. in $D^*(\mathcal{A})$

$\text{obj}\, D^*(\mathcal{A}) = \text{obj}\, C^*(\mathcal{A}) = \text{obj}\, K^*(\mathcal{A})$.

$\text{Hom}_{D^*(\mathcal{A})}(X, Y) = \{f/s \text{ or } X \overset{s}{\Leftarrow} Z \overset{f}{\rightarrow} Y\}$.

Fix $f \in \text{Hom}_{K^*(\mathcal{A})}(X, Y)$.

$F^*(f) = f/\text{id}_X = X \overset{\text{id}_X}{\Leftarrow} X \overset{f}{\longrightarrow} Y$.

$F^*(f)$ is an isom $\Leftrightarrow$ f is an quasi-isom.

$F^*(f) = 0 \Leftrightarrow \exists\, g \in Q$, s.t. $fg \sim 0$, (or $\exists\, s \in Q$ st. $sf \sim 0$)

$F^*(X)\ (=X) = 0 \Leftrightarrow X$ is acyclic in $C^*(\mathcal{A}) \Leftrightarrow X$ is an exact seq.

Note that: $F^*(f) = 0 \not\Rightarrow f \sim 0$. $(fg = 0, g \in Q, 0 = F^*(fg) = F^*(f)\underbrace{F^*(g)}_{\text{isom}} \Rightarrow F^*(f) = 0)$.

E.g. $X: \to 0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0 \to \cdots$ $f = \text{id}_X$, $g: X \to 0$.

Since X is exact, g is an quasi-isom, $fg = 0 \Rightarrow F(f) = 0$.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] \arrow[dl, "0"'] & \mathbb{Z} \arrow[r] \arrow[d] \arrow[dl, "\times\frac{1}{2}"'] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

$f \not\sim 0$

So, the following relations are NOT invertible.

$f = g$ in $C^*(\mathcal{A}) \Rightarrow f = g$ in $K^*(\mathcal{A}) \Rightarrow f = g$ in $D^*(\mathcal{A})$

$\Rightarrow H^i(f) = H^i(g)$

D.T. in $D^*(\mathcal{A})$ are triangles isom to "tri induced by mapping cones" in $D^*(\mathcal{A})$. (and also isom to "tri induced by mapping cylinders")

The following prop says D.T. in $D^*(\mathcal{A})$ are all isom to triangles induced by s.e.s. of cpx. and vice versa.

(Recall in $K^*(\mathcal{A})$ c.s.s.e.s of cpx)

Prop 5.1.2. Suppose $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $C(\mathcal{A})$ then $\exists$ mor in $D^*(\mathcal{A})$ $h: Z \to X[1]$ s.t. $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a D.T. in $D^*(\mathcal{A})$. Conversely, every D.T. in $D^*(\mathcal{A})$ is isom to such a triangle.

Pf: By prop 2.7.2, we have

$$\begin{tikzcd}
0 \arrow[r] & \underline{X} \arrow[r] \arrow[d, equal] & \underline{Cyl(f)} \arrow[r] \arrow[d, "s"] & \underline{Cone(f)} \arrow[r] \arrow[d, "t"] & 0 \\
0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$
quasi-isom in $K^*(\mathcal{A})$

$\downarrow$

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & Cyl(f) \arrow[r] \arrow[d, "\simeq", "s"'] & Cone(f) \arrow[r, "w"] \arrow[d, "\simeq", "t"'] & X[1] \arrow[d, equal] \\
X \arrow[r] & Y \arrow[r] & Z \arrow[r, "h := wt^{-1}"] \arrow[u, bend right, "t^{-1}"'] & X[1]
\end{tikzcd}$$
isom in $D^*(\mathcal{A})$

the first row is a D.T. in $D^*(\mathcal{A})$ $\Rightarrow$ the second row is also a D.T. in $D^*(\mathcal{A})$.

Conversely:

D.T. in $D^*(\mathcal{A})$ $\simeq$ D.T. induced by a mapping cone $\simeq$ D.T. induced by a $\underline{\text{c.s.s.e.s.}}$ $\Rightarrow$ s.e.s. $\square$.

Prop 5.1.3. Let $\alpha = a/s \in \mathrm{Hom}_{D^*(\mathcal{A})}(X, Y)$, $s \in Q$. TFAT:

(1). α is an isom $\Leftrightarrow$ $a \in Q$

(2) (cohom obj is an invariant under isom. i.e. if α is an isom, $H^i(X) \simeq H^i(Y)$, $\forall i$.

(3) $\alpha = 0 \Leftrightarrow \exists t \in Q$ s.t. $at \sim 0$.

Pf: (1). $\alpha = a/s = a/1 \circ \underset{\text{isom}}{\boxed{1/s}}$. So α is an isom $\Leftrightarrow$ $a/1$ is an isom $\Leftrightarrow$ $a \in Q$. $\square$.

(2). $\underline{\alpha : X \overset{s}{\Leftarrow} Z \overset{a}{\Rightarrow} Y}$. If α is an isom, by (1), $a \in Q$.

$\downarrow H^i(-)$

$H^i(\alpha) : H^i(X) \xleftarrow{H^i(s)} H^i(Z) \xrightarrow{H^i(a)} H^i(Y)$.

$s, a \in Q \Rightarrow H^i(s)$ & $H^i(a)$ are isom

$\Rightarrow H^i(X) \simeq H^i(Y)$, $\forall i$.

(3). $0 = \alpha = a/s = a/1 \circ \underbrace{1/s}_{\text{isom}} \Leftrightarrow a/1 = 0$.

$$\Leftrightarrow \begin{tikzcd}
 & Z \arrow[dl, "id"'] \arrow[dr, "a"] & \\
Z & K \arrow[l, "t"'] \arrow[u] \arrow[d] & Y \\
 & Z \arrow[ul, "id"] \arrow[ur, "0"'] &
\end{tikzcd} \Rightarrow at \sim 0. \quad \square$$

Fundament thm in $D^*(\mathcal{A})$

Thm 5.1.4. $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ D.T. in $D^*(\mathcal{A})$. Then we have a l.e.s.

$$\cdots \to H^n(X) \xrightarrow{H^n(u)} H^n(Y) \xrightarrow{H^n(v)} H^n(Z) \xrightarrow{H^n(w)} H^n(X[1]) \to \cdots$$

& Mor of D.T. induce the mor of l.e.s.

Pf:
$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d] & Y \arrow[r, "v"] \arrow[d] & Z \arrow[r, "w"] \arrow[d] & X[1] \arrow[d] \\ X' \arrow[r, "u'"] & Cyl(u') \arrow[r] & Cone(u') \arrow[r] & X'[1] \end{tikzcd}$$

$\mathcal{A} \hookrightarrow D(\mathcal{A})$

Def: embedding functor: $C_0 : \mathcal{A} \longrightarrow K(\mathcal{A})$, 0-th

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, "0"] & \cdots \\ Y & \cdots \arrow[r] & 0 \arrow[r] & Y \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

Def: $D_0 : \mathcal{A} \xrightarrow{C_0} K(\mathcal{A}) \xrightarrow{F} D(\mathcal{A})$.

We also use X to denote $C_0(X)$, $D_0(X)$.

Prop 5.1.5. D_0 is fully faithful.

Pf: Fix $X, Y \in \mathcal{A}$, $D_0 : \mathrm{Hom}_{\mathcal{A}}(X,Y) \longrightarrow \underline{\mathrm{Hom}_{D(\mathcal{A})}(X,Y)}$. To show D_0 is bijective.

"inj". Suppose $f \in \mathrm{Hom}_{\mathcal{A}}(X,Y)$, $D_0(f) = 0 \Rightarrow F(C_0(f)) = 0$

$\Rightarrow \exists\, t \in Q$ s.t. $\underline{C_0(f)\, t \sim 0} \Rightarrow H^0(C_0(f))\, H^0(t) = 0$ in $\mathrm{Hom}_{\mathcal{A}}$ $\Rightarrow H^0(C_0(f)) = 0$.

isom

$\Rightarrow f = 0$.

$$\begin{tikzcd} & \cdots \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, "0"] & \cdots & C_0(f) \\ & \cdots \arrow[r] & 0 \arrow[r] & Y \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

$X \overset{id_X}{\cong} X$, $Y \overset{id_Y}{\cong} Y$

"surj". Fix $\alpha = a/s \in \mathrm{Hom}_{D(\mathcal{A})}(D_0(X), D_0(Y))$ i.e.

$$\alpha : D_0(X) \xleftarrow{s \in Q} Z \xrightarrow{a} D_0(Y)$$

Consider: $H^0(s) : H^0(Z) \longrightarrow \underbrace{H^0(D_0(X))}_{=X}$

$H^0(a) : H^0(Z) \longrightarrow \underbrace{H^0(D_0(Y))}_{=Y}$

Let $\underline{u} = H^0(a) \circ H^0(s)^{-1} : X \longrightarrow Y \in \mathrm{Hom}_{\mathcal{A}}(X,Y)$.

$(\underline{D_0(u) = \alpha})$.

Consider $\tau_{\leq 0} Z : \; \to Z^{-2} \xrightarrow{d^{-2}} Z^{-1} \xrightarrow{d^{-1}} \ker d^0 \to 0 \to \cdots$

Let $i : \tau_{\leq 0} Z \to Z$, then we have

$$\begin{tikzcd} \tau_{\leq 0} Z \arrow[r, "i"] \arrow[d] & Z \arrow[d, "s"] \\ {} & {} \end{tikzcd}$$ in $C(\mathcal{A})$, $H^0(s)$

0-th

$$\begin{tikzcd} \ker d^0 \arrow[r, "i"] \arrow[d, "\pi"] & Z^0 \arrow[d, "s"] \\ {} & {} \end{tikzcd}$$

$$H^0(s) : 0 \to \mathrm{Im}\, d^{-1} \xrightarrow{\beta} \ker d^0 \xrightarrow{\pi} H^0(Z) \to 0.$$

X, $\varphi \circ s \circ i$, $H^0(s)$, $s \circ i$

$$\begin{tikzcd} \cdot \arrow[d] \arrow[r, phantom, "\circlearrowright"] & \cdot \arrow[d, "s"] \\ H^0(t) \arrow[r, "H^0(s)"] & X \end{tikzcd} \quad \text{in } C(\mathcal{A}).$$

stalk cpx

$$\begin{tikzcd} \cdot \arrow[d, "\pi"] \arrow[r, phantom, "\circlearrowright"] & \cdot \arrow[d, "s"] \\ H^0(t) \arrow[r, "H^0(s)"] & X \end{tikzcd}$$

$\ker d^0 / \operatorname{Im} d^{-1}$

$H^0(s)\circ\pi = \varphi = s\circ i.$

$H(s)$: $0 \to \operatorname{Im} d^{-1} \xrightarrow{i} \ker d^0 \xrightarrow{\pi} H^0(Z) \to 0.$

$$\begin{tikzcd} \operatorname{Im} d^{-1} \arrow[r, "\beta"] & \ker d^0 \arrow[d, "i"] & & \\ Z^{-1} \arrow[u, "\alpha"] \arrow[r, "d^{-1}"] \arrow[d] & Z^0 \arrow[r, "d^0"] \arrow[d, "\varphi"] \arrow[d, bend left, "s"] & Z^1 \\ 0 \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

$a\circ i$ $\quad H^0(a)$

$\Rightarrow \varphi = s\circ i.$

$\varphi\circ\beta\circ\alpha = s\circ i\circ\beta\circ\alpha = s\circ d^{-1} = 0.$

epi

$\Rightarrow \varphi\circ\beta = 0.$

$\boxed{D_0(u) = u/Id = a/s.}$

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "a"] & \\ X & \tau_{\le 0}Z \arrow[l, "s\circ i"] \arrow[r, "a\circ i"] \arrow[u] \arrow[d, "s\circ i"] & Y \\ & X \arrow[ul, "id_X"] \arrow[ur, "u = L_0(u)"'] & \end{tikzcd}$$

$\circlearrowright$: trivial.

$\circlearrowright$: $u\circ s\circ i = H^0(a)\circ H^0(s)^{-1}\circ s\circ i$
$= H^0(a)\circ \cancel{H^0(s)^{-1}}\circ \cancel{H^0(s)}\circ\pi$
$= H^0(a)\circ\pi$
$= a\circ i$

$\Rightarrow F(L_0(u)) = u/id_X = a/s = \alpha.$

$\quad = D_0(u).$

$\Rightarrow$ "surj".

Now we can view $X\in\mathcal{A}$ as an obj in $D(\mathcal{A})$.

E.g. 5.1.6. $X\in\mathcal{A}$, $D(\mathcal{A})$. $\cdots \to P_1 \to P_0 \xrightarrow{\varepsilon} X \to 0$. proj resol of X.

Let $P: \cdots \to P_1 \to P_0 \to 0$. Then $X \simeq P$ in $D(\mathcal{A})$.

In fact, $\varepsilon: P\to X$ is an quasi-isom in $K(\mathcal{A})$.

Lem 5.1.7. $\mathcal{C}$: Add cat, $\mathcal{D}$: full sub cat of $\mathcal{C}$. S: multi sys of $\mathcal{C}$. Suppose $S\cap\mathcal{D}$ is a multi sys of $\mathcal{D}$ and

(i) Let $s: X\to X'$, $s\in S$, $X\in\mathcal{D}$, $\exists f: X'\to X''$ s.t. $X''\in\mathcal{D}$ and $fs\in S$.

$$\underbrace{\boxed{X} \xrightarrow{s} X' \xrightarrow{f} X''}_{\in S} \qquad X\in\mathcal{D},\ X''\in\mathcal{D}.$$

or (ii) Let $s: X'\to X$, $s\in S$, $X\in\mathcal{D}$. $\exists f: X''\to X'$ s.t. $X''\in\mathcal{D}$ and $sf\in S$.

Then $\underline{(S\cap\mathcal{D})^{-1}\mathcal{D} \longrightarrow S^{-1}\mathcal{C}}$ is fully-faithful, i.e. $(S\cap\mathcal{D})^{-1}\mathcal{D}$ is a full sub cat of $S^{-1}\mathcal{C}$.

Pf: (ii). $\mathcal{D} \longrightarrow \mathcal{C}$ embedding functor

Hom $\longrightarrow$ Hom.

$\underline{\mathrm{Hom}[(S\cap\mathcal{D})^{-1}]}$ $\quad$ $\mathrm{Hom}[S^{-1}]$

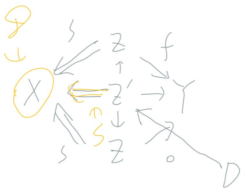
$\cdot \xleftarrow{} \mathcal{D} \xrightarrow{f} \cdot \quad \longmapsto \quad \cdot \xleftarrow{} S \xrightarrow{f} \cdot$

Fix $X, Y \in \mathcal{D}$.

$$\mathrm{Hom}_{(S\cap\mathcal{D})^{-1}\mathcal{D}}(X,Y) \xrightarrow{F} \mathrm{Hom}_{S^{-1}\mathcal{C}}(X,Y).$$
$$f/s \longmapsto f/s.$$

"inj". i.e. $f/s = 0$ in $S^{-1}\mathcal{C} \Rightarrow f/s = 0$ in $(S\cap\mathcal{D})^{-1}\mathcal{D}$.

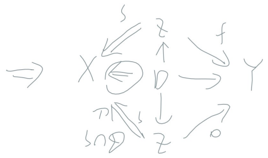
suppose $f/s = 0$ in $S^{-1}\mathcal{C}$.



By (ii), $\exists\, D \in \mathcal{D}$

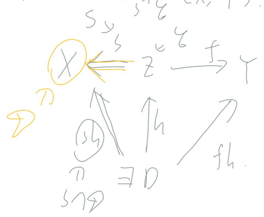
$D \longrightarrow Z'$

s.t. $\underset{\mathcal{D}}{\underset{\cap}{D}} \to Z' \Longrightarrow \underset{\mathcal{D}}{\underset{\cap}{X}} \in S\cap\mathcal{D}$



$\Rightarrow f/s = 0$ in $(S\cap\mathcal{D})^{-1}\mathcal{D}$.

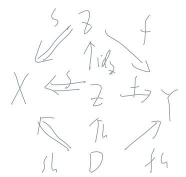
"surj". Fix $f/s \in \mathrm{Hom}_{S^{-1}\mathcal{C}}(X,Y)$.



We obtain a mor $X \xLeftarrow{sh} D \xrightarrow{fh} Y \in \mathrm{Hom}_{(S\cap\mathcal{D})^{-1}\mathcal{D}}(X,Y)$.

View $X \xLeftarrow{sh} D \xrightarrow{fh} Y$ as a mor in $\mathrm{Hom}_{S^{-1}\mathcal{C}}(X,Y)$

It's easy to see:



$\Rightarrow F(fh/sh) = f/s \Rightarrow$ surj.

Lem 5.1.8. (1). $\mathcal{C} = K^-(\mathcal{A})$, $\mathcal{D} = K^b(\mathcal{A})$, $S = Q_-$. Then condition (i) holds.

(2). $\mathcal{C} = K(\mathcal{A})$, $\mathcal{D} = K^-(\mathcal{A})$, $S = Q$. Then condition (ii) holds.

Pf: (1). $X' \in K^-(\mathcal{A})$, $X \in K^b(\mathcal{A})$.

W.l.o.g. suppose $X'^n = 0$, $\forall n > 0$.

Since we have $X \longrightarrow X'$ quasi-isom and $X \in K^b(\mathcal{A})$, suppose $H^n(X') = 0$, $\forall n < -m$.

Def: $\sigma: X' \longrightarrow X''$

$H^n(X')=0, \forall n<-m$.

Def: $\partial: X' \to X''$

$$\begin{array}{ccccccccccccc}
X' & & \to X'^{-(m+1)} & \to & X'^{-m} & \to & X'^{-(m-1)} & \to \cdots \to & X'^{0} & \to 0 \to \cdots \\
\partial\downarrow & & \downarrow & & \downarrow & & \| & & \| & \\
X'' & & \to 0 & \to & \mathrm{Ind}^{-m} & \to & X'^{-(m-1)} & \to \cdots \to & X'^{0} & \to 0 \to
\end{array}$$

$X'' \in K^b(A)$.

Obviously: ∂ is a quasi-iso.

(2). $X' \in K(A)$, $X \in K^-(A)$.

Since $X' \to X$ quasi-iso, Suppose $H^n(X')=0, \forall n \geq 0$.

Def: $\partial: X'' \to X'$

$X'' \in K^-(A)$.

$$\begin{array}{ccccccc}
X'' & \cdots \to & X'^{-1} & \to & \mathrm{Ind}^{-1} & \to & 0 \to \cdots \\
\downarrow\partial & & \| & & \downarrow & & \downarrow \\
X' & \cdots \to & X'^{-1} & \to & X'^{0} & \to & X'^{1} \to \cdots
\end{array}$$

Obviously: $\partial \in Q$. $\square$

By Lem 5.1.7 & 5.1.6.

$D^b(A)$ is a full sub cat of $D^-(A)$.

$D^-(A)$ $\qquad\qquad$ $D(A)$.

$\Rightarrow$ $D^b(A)$ is a full subcat of $D(A)$. Hence tri subcat.

In particular, $X, Y \in D^b(A)$, $X \simeq Z$, $Y \simeq Z$ in $D(A)$ $\Rightarrow$ $X \simeq Y$ in $D^b(A)$.

Cor 5.1.9. $D^b(A)$, $D^-(A)$, $D^+(A)$ are all tri subcat of $D(A)$. and $D^+(A) \cap D^-(A) = D^b(A)$.

Here, when we say $D^b(A)$ is tri subcat of $D(A)$. The obj of $D^b(A)$ is those cpx isom to $D^b(A)$. Similar to $D^{-,+}(A)$.

Useful property.

Lem 5.1.10 (1). If A has enough proj obj. P: upper bounded proj cpx. $Y \in C(A)$. Then $F: f \mapsto F(f) = f/\mathrm{Id}_P$ is an isom of add grp.

$$\mathrm{Hom}_{K(A)}(P,Y) \simeq \mathrm{Hom}_{D(A)}(P,Y).$$

In particular, if Y upper-bounded, then

$$\mathrm{Hom}_{K^-(A)}(P,Y) \simeq \mathrm{Hom}_{D^-(A)}(P,Y).$$

(2) If A has enough inj obj, I: lower bounded inj cpx, $X \in C(A)$. Then F':

(2) If $\mathcal{A}$ has enough inj obj, I lower bounded inj cpx, $X \in C(\mathcal{A})$. Then $F': f \mapsto F'(f) = \mathrm{Id}_I \backslash f$ is an isom of add grp

$$\mathrm{Hom}_{K(\mathcal{A})}(X, I) \simeq \mathrm{Hom}_{D(\mathcal{A})}(X, I).$$

In particular, if X lower bounded, then

$$\mathrm{Hom}_{K^-(\mathcal{A})}(X, I) \simeq \mathrm{Hom}_{D(\mathcal{A})}(X, I).$$

Pf: (1). "F is inj". If $F(f) = f/\mathrm{Id} = 0$, then $\exists\, \underline{t^{qis}: X \to P}$ s.t. $ft \sim 0$. By Cor 4.2.5 (If P upper bounded proj cpx, $t: X \to P$ ^{qis} then t is split epi. i.e. $\exists\, t'^{qis}: P \to X$ s.t. $tt' \sim \mathrm{Id}_P$).

$\exists\, g^{qis}: P \to X$ s.t. $tg \sim \mathrm{Id}_P$. Then

$$f \sim f\,\mathrm{Id}_P \sim \underline{ft}g \sim 0.$$

$\Rightarrow F$ is inj.

(2). "surj".

Fix $f/s \in \mathrm{Hom}_{D(\mathcal{A})}(P, Y)$, $P \overset{s}{\Leftarrow} X \xrightarrow{f} Y$. By Cor 4.2.5

$\exists\, g^{qis}: P \to X$ s.t. $sg \sim \mathrm{Id}_P$. Then

$$f/s = fg/\underbrace{sg}_{\simeq} = fg/\mathrm{Id}_P = F(fg).$$

$\Rightarrow F$ is surj. $\square$

§5.2. upper/lower bounded derived cat as htp cat.

Thm 5.2.1 (1). Suppose $\mathcal{A}$ has enough proj obj, $P = \{\text{proj obj}\}$. Then the natural functor induce two tri equiv.

$$D^-(\mathcal{A}) \simeq K^-(P), \quad \underline{D^b(\mathcal{A}) \simeq K^{-,b}(P)} \text{ has only fin. nonzero cohomo.}$$

In particular, if every obj in $\mathcal{A}$ has fin proj dim, then $D^b(\mathcal{A}) \simeq K^b(P)$.

(1') Suppose $\mathcal{A}$ has enough inj obj, $I = \{\text{inj obj}\}$. Then the natural functor induce two tri equiv.

$$D^+(\mathcal{A}) \simeq K^+(I), \quad D^b(\mathcal{A}) \simeq K^{+,b}(I).$$

In particular, if every obj in $\mathcal{A}$ has fin inj dim, then $D^b(\mathcal{A}) \simeq K^b(I)$.

Pf: (1). $\underbrace{K^-(P) \hookrightarrow K^-(\mathcal{A}) \xrightarrow{F} D^-(\mathcal{A})}_{G:}$. By Lem 5.1.10. it is fully-faithful. Only need to show it is essentially full.

By thm 4.2.1. $\forall X \in D^-(\mathcal{A})$, $\exists\, \rho X \in K^-(P)$ & quasi-isom $\rho X \to X$. That is to say, G is essentially full.

... $\forall X \in D^-(\mathcal{A})$, $\exists\, \rho X \in K^-(\mathcal{P})$ & quasi-iso $\rho X \to X$. That is to say, G is essentially full.

. similar to the second tri equiv. $\square$.

Thm 5.2.3 Suppose $\mathcal{A}$ has enough proj obj (or enough inj obj), $M, N \in \mathcal{A}$. Then $\mathrm{Ext}^n_{\mathcal{A}}(M,N) \simeq \mathrm{Hom}_{D^b(\mathcal{A})}(M, N[n])$

Pf: Let $\to P^{-1} \to P^0 \to M \to 0$ be proj resol of M.

$P := \to P^{-1} \to P^0 \to 0$. We know $M \simeq P$ in $K^-(\mathcal{A})$. Then

$$\mathrm{Hom}_{D^b(\mathcal{A})}(M, N[n]) = \mathrm{Hom}_{D^-(\mathcal{A})}(M, N[n])$$

$$= \mathrm{Hom}_{D^-(\mathcal{A})}(P, N[n]).$$

$$\overset{\text{By Lem 5.1.10}}{\simeq} \mathrm{Hom}_{K^-(\mathcal{A})}(P, N[n]).$$

By prop 2.7.1, $\mathrm{Hom}_{K^-(\mathcal{A})}(P, N[n]) = H^n \mathrm{Hom}^\bullet(P, N)$. Since N is a stalk cpx, $\mathrm{Hom}^\bullet(P,N)$ can be obtained by act $\mathrm{Hom}(-,N)$ on P.

So $H^n\mathrm{Hom}^\bullet(P,N) = \underline{H^n \mathrm{Hom}_{\mathcal{A}}(P,N)} = \mathrm{Ext}^n_{\mathcal{A}}(M,N)$ $\square$.

Rmk 5.2.4. The iso above can be defined explicitly.

$$\mathrm{Ext}^n_{\mathcal{A}}(M,N) \xrightarrow{F} \mathrm{Hom}_{D^b(\mathcal{A})}(M, N[n]).$$

$$\bar{h} \longmapsto \underline{M \overset{s}{\Leftarrow} P \xrightarrow{\tilde{h}} N[n]}$$

$$\mathrm{Hom}(P^{-(n-1)}, N) \xrightarrow{\mathrm{Hom}(d_P^{-(n-1)},N)} \mathrm{Hom}(P^{-n}, N) \longrightarrow \mathrm{Hom}(P^{-(n+1)}, N)$$

$-n$-th

$h \in \mathrm{Hom}(P^{-n}, N)$

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\tilde{h}"'] & P^{-n} \arrow[r] \arrow[d, "h-h'"] \arrow[dl, "0"'] & P^{-(n-1)} \arrow[d] \arrow[dl] \\ 0 \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

"F is well-defined".

If $h, h' \in \bar{h}$, or $\underline{h - h' \in \mathrm{Im}\,\mathrm{Hom}(d_P^{-(n-1)}, N)}$.

$\Longleftrightarrow h - h' \sim 0.$

$\Rightarrow h = h'$ in $K(\mathcal{A})$.

$\Rightarrow \tilde{h} = \tilde{h'} \Rightarrow M \overset{s}{\Leftarrow} P \xrightarrow{\tilde{h}} N[n]$

$\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\qquad\qquad\qquad\qquad M \overset{s}{\Leftarrow} P \xrightarrow{\tilde{h'}} N[n]$

"F is inj". If $h, h' \in \mathrm{Ext}^n_{\mathcal{A}}(M,N)$, suppose $\tilde{h}/s = \tilde{h'}/s$

$\Rightarrow (\tilde{h} - \tilde{h'})/s = 0. \Rightarrow (\tilde{h} - \tilde{h'})/1 = 0.$

$$\Rightarrow \begin{tikzcd} & P \arrow[dl, "id"'] \arrow[dr, "\tilde{h}-\tilde{h'}"] & \\ P & Z \arrow[l, "t"'] \arrow[u, shift left, "t"] \arrow[d, shift right, "t"'] \arrow[r, phantom, "\circlearrowright"] & N[n] \\ & P \arrow[ul, "id"] \arrow[ur, "0"'] & \end{tikzcd}$$

in $K(\mathcal{A})$.

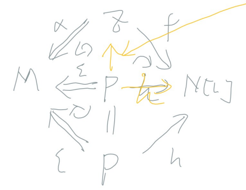
By Cor 4.2.5: $\exists\, s: P \to Z$ s.t. $ts = id_P$.

"$\circlearrowright$" $(\tilde{h} - \tilde{h'})t = 0.$

$(\tilde{h} - \tilde{h'})\underbrace{ts}_{= id_P} = 0 \Rightarrow \tilde{h} = \tilde{h'}$, in $K(\mathcal{A})$.

$\Rightarrow$ "F is inj".

"F is surj". Fix $M \xleftarrow{\alpha} P \xrightarrow{f} N[1]$, (hope to find h s.t. $h/\xi = f/\xi$).



By Lem 4.2.4, $\mathrm{Hom}(P, \xi)$ is an isom,

Then we find h s.t. $F(h) = f/\alpha$.

$\Rightarrow$ "F is surj".

E.g. 5.2.5. $0 \to N \xrightarrow{f} L \xrightarrow{g} M \to 0$, h is the corresponding element in $\mathrm{Ext}^1_{\mathcal{A}}(M, N)$, h also denote the correspond element in $\mathrm{Hom}_{D^b}(M, N[1])$. Then $M \xrightarrow{f} L \xrightarrow{g} M \xrightarrow{h} N[1]$ is a D.T. in $D^b(\mathcal{A})$.

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] & 0 & \\
0 \arrow[r] & N \arrow[u, "id_N"] \arrow[r, "f"] & L \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] & L \arrow[r] & M \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[u, equal] \arrow[r] & \mathrm{Cyl} \arrow[u] \arrow[r] & \mathrm{Cone} \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

$\longrightarrow \cdot \xLeftarrow{g} M$

$\longrightarrow \cdot \xLeftarrow{id_N} N[1]$

$X \longrightarrow \cdot \Longleftarrow Y.$